Solution of HW6

3 Since the Poisson process $X(s)$ has the independent and stationary increments, we have

$$\begin{aligned} P(X(s)=m \mid X(t)=n) &= \frac{P(X(s)=m, X(t)=n)}{P(X(t)=n)} \\ &= \frac{P(X(t)-X(s)=n-m)P(X(s)=m)}{P(X(t)=n)} \\ &= \frac{\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-m}}{(n-m)!} \cdot \frac{e^{-\lambda s}(\lambda s)^m}{m!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \\ &= \binom{n}{m}\left(\frac{s}{t}\right)^m\left(1-\frac{s}{t}\right)^{n-m} \sim B\left(n, \frac{s}{t}\right). \end{aligned}$$

4 For $t \geq 0$, it is easy to see that two sets are equal: $\{T_m \leq t\} = \{X(t) \geq m\}$. Hence,

$$\begin{aligned} F_{T_m}(t) = P(T_m \leq t) = P(X(t) \geq m) & \\ &= 1 - \sum_{k=0}^{m-1} P(X(t)=k) \\ &= 1 - \sum_{k=0}^{m-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}. \end{aligned}$$

For $t < 0$, as T_m is non-negative random variable so $\{T_m \leq t\}$ is empty and hence $F_{T_m}(t) = 0$.

5 Differentiate F_{T_m} to obtain the density of T_m. For $t < 0$, we simply have $f_{T_m}(t) = 0$. For $t \geq 0$, we have

$$f_{T_m}(t) = e^{-\lambda t}\left(\sum_{k=1}^{m-1} \frac{\lambda^k t^{k-1}}{(k-1)!} - \lambda \sum_{k=0}^{m-1} \frac{\lambda^k t^k}{k!}\right) = \frac{\lambda^m t^{m-1} e^{-\lambda t}}{(m-1)!}.$$

(It is of Gamma distribution of parameter m and λ.)

6 We calculate directly that

$$\begin{aligned} P(T_1 \leq s \mid X(t)=n) &= P(X(s) \geq 1 \mid X(t)=n) \\ &= 1 - P(X(s)=0 \mid X(t)=n) \\ &= 1 - \binom{n}{0}\left(\frac{s}{t}\right)^0\left(1-\frac{s}{t}\right)^n \quad \text{(by Q3)} \\ &= 1 - \left(1-\frac{s}{t}\right)^n. \end{aligned}$$

SQ1 The probablity density function of an exponential distribution is

$$f_\lambda(t) = \begin{cases} \lambda e^{-\lambda t} & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

with expected value $\frac{1}{\lambda}$. So $\lambda = \frac{1}{2}$ in this problem.

(a)

$$P(t \geqslant 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}t} = \frac{1}{e}.$$

(b)

$$P(t \geqslant 5 | t \geqslant 3) = P(t \geqslant 2) = \frac{1}{e}.$$

SQ2 Let $X \sim \text{Exp}(\frac{1}{20}), Y \sim \text{Exp}(\frac{1}{30})$ denote the time that Alice and Betty are done respectively.

(a)

$$P\{X < Y\} = P\{X = \min(X, Y)\} = \frac{\frac{1}{20}}{\frac{1}{20} + \frac{1}{20}} = \frac{3}{5}.$$

(b) Note that $\min(X, Y) \sim \text{Exp}(\frac{1}{20} + \frac{1}{30})$.

$$E[\max(X, Y)] = E[X + Y - \min(X, y)] = 20 + 30 - \frac{1}{\frac{1}{20} + \frac{1}{30}} = 38.$$

SQ3 Ron: $X_1 \sim \text{Exp}(1)$, Sue: $X_2 \sim \text{Exp}(2)$, Ted: $X_3 \sim \text{Exp}(3)$.

(a) We may condition on who is the first one to leave. For example, if Ron is the first one to leave, then the expected time until only one student remains is $\min\{X_2, X_3\}$. Let F_i be the event that $\{X_i = \min\{X_1, X_2, X_3\}\}$. Then, the expected time is given by

$$E[\min\{X_2, X_3\} \mid F_1] P(F_1) + E[\min\{X_3, X_1\} \mid F_2] P(F_2) + E[\min\{X_1, X_2\} \mid F_3] P(F_3)$$

As an example, we may calculate $E[\min\{X_2, X_3\} \mid F_1] P(F_1)$. Note that for $t > 0$,

$$
\begin{aligned}
E\left[\min\left\{X_2, X_3\right\} \mid F_1\right] P\left(F_1\right) &= \frac{1}{P\left(F_1\right)} P\left(X_1 < \min\left\{X_2, X_3\right\} < t\right) \\
&= \frac{1}{P\left(F_1\right)} \int_0^t P\left(X_1 < s\right) f_{\min\{X_2,X_3\}}(s) ds \\
&= \frac{1}{P\left(F_1\right)} \int_0^t \left(1 - e^{-\lambda_1 s}\right) \left(\lambda_2 + \lambda_3\right) e^{-(\lambda_2+\lambda_3)s} ds \\
&= \frac{\lambda_2 + \lambda_3}{P\left(F_1\right)} \int_0^t e^{-(\lambda_2+\lambda_3)s} - e^{-(\lambda_1+\lambda_2+\lambda_3)s} ds \\
&= 1 - \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1} e^{-(\lambda_2+\lambda_3)t} + \frac{\lambda_2 + \lambda_3}{\lambda_1} e^{-(\lambda_1+\lambda_2+\lambda_3)t}
\end{aligned}
$$

The last step uses the fact that $P\left(F_1\right) = \lambda_1 / \left(\lambda_1 + \lambda_2 + \lambda_3\right)$. One may calculate the average time by the formula

$$
E\left[\min\left\{X_2, X_3\right\} \mid F_1\right] = \int_0^\infty P\left(\min\left\{X_2, X_3\right\} > t \mid F_1\right) dt
$$

This is legitimate only when the random variable X satisfies $X \geq 0$. Or one can first find the probability density function $f_X(t)$ and integrate $\int_0^\infty t f(t) dt$. Hence,

$$
\begin{aligned}
E\left[\min\left\{X_2, X_3\right\} \mid F_1\right] &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\lambda_1\left(\lambda_2 + \lambda_3\right)} - \frac{\lambda_2 + \lambda_3}{\lambda_1\left(\lambda_1 + \lambda_2 + \lambda_3\right)} \\
&= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} = \frac{11}{30}
\end{aligned}
$$

Using the formula above, one can show that $E\left[\min\left\{X_3, X_1\right\} \mid F_2\right] = 5/12$ and $E\left[\min\left\{X_1, X_2\right\} \mid F_3\right] = 5/12$. Therefore, the required expected amount of time is

$$
\frac{11}{30} \cdot \frac{1}{6} + \frac{5}{12} \cdot \frac{2}{6} + \frac{1}{2} \cdot \frac{3}{6} = \frac{9}{20}
$$

(b) For Ron

$$
\begin{aligned}
P(X_1 > X_2, X_1 > X_3) =& 1 - P(X_1 \leqslant X_2 \text{ or } X_1 \leqslant X_3) \\
=& 1 - (P(X_1 \leqslant X_2) + P(X_1 \leqslant X_3) - P(X_1 \leqslant X_2, X_1 \leqslant X_3)) \\
=& 1 - (\frac{1}{3} + \frac{1}{4} - \frac{1}{6}) \\
=& \frac{7}{12}.
\end{aligned}
$$

For Sue,

$$
\begin{aligned}
P(X_2 > X_1, X_2 > X_3) =& 1 - P(X_2 \leqslant X_1 \text{ or } X_2 \leqslant X_3) \\
=& 1 - (P(X_2 \leqslant X_1) + P(X_2 \leqslant X_3) - P(X_2 \leqslant X_1, X_2 \leqslant X_3)) \\
=& 1 - (\frac{2}{3} + \frac{2}{5} - \frac{2}{6}) \\
=& \frac{4}{15}.
\end{aligned}
$$

For Ted,

$$
\begin{aligned}
P(X_3 > X_2, X_3 > X_1) =& 1 - P(X_3 \leqslant X_2 \text{ or } X_3 \leqslant X_1) \\
=& 1 - (P(X_3 \leqslant X_2) + P(X_3 \leqslant X_1) - P(X_3 \leqslant X_2, X_3 \leqslant X_1)) \\
=& 1 - (\frac{3}{5} + \frac{3}{4} - \frac{3}{6}) \\
=& \frac{3}{20}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
E[\max(X_1, X_2, X_3)] =& E[X_1 + X_2 + X_3 - \min(X_1, X_2) - min(X_1, X_3) - \min(X_2, X_3) + \mathrm{m} \\
=& 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \\
=& \frac{73}{60}.
\end{aligned}
$$

SQ4 The rate matrix D is

$$
D = \begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} -1/3 & 1/3 & 0 \\ 0 & -1/4 & 1/4 \\ 1 & 0 & -1 \end{pmatrix}
\end{array}
$$

and the Markov matrix Q is

$$
Q = \begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\end{array}
$$

SQ5 Let $X(t)$ be the number of machine in operations. Then $\mathcal{S} = \{0, 1, 2, 3\}$. For $x \in \{0, 1, 2\}$, as there is one repairman to repair a machine, we have $q_{x,x+1} = 1/4$. Denote the breakdown time of each machine to be Y_1, Y_2 and Y_3 respectively. Then, each $Y_i \sim Exp(1/60)$ for $i = 1, 2$ and 3. Note $\min\{Y_1, Y_2\} \sim Exp(1/60 + 1/60)$ and $\min\{Y_1, Y_2, Y_3\} \sim Exp(1/60 + 1/60 + 1/60)$. For $x \in \{1, 2, 3\}$, $q_{x,x-1} = 1/E(\min_{k=1,\ldots x} Y_k) = x/60$ Therefore, we can write

The rate matrix D is

$$
D = \begin{array}{c}
\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{pmatrix} -1/4 & 1/4 & 0 & 0 \\ 1/60 & -4/15 & 1/4 & 0 \\ 0 & 1/30 & -17/60 & 1/4 \\ 0 & 0 & 1/20 & -1/20 \end{pmatrix}
\end{array}
$$

and the Markov matrix Q is

$$Q = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/16 & 0 & 15/16 & 0 \\ 0 & 2/17 & 0 & 15/17 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$